# METHOD OF ORDERING THE VALUES OF A LINEAR FUNCTION ON A SET OF PERMUTATIONS

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*The paper deals with a new method of solving a combinatorial problem with account for the properties of the set of permutations and its structure. Using this method, the values of the linear objective function are sequenced and the set of permutations is decomposed over hyperplanes, with account of element recurrences. This makes it possible to develop an algorithm of finding the point (an element of the set of permutations) at which the objective function attains a given value.*

**Keywords:** *combinatorial sets, set of permutations, permutation polyhedron, graphs, hyperfaces, Hamiltonian path.*

## INTRODUCTION

Many operations research problems such as plant scheduling, resource allocation, control, and network planning can be described by discrete optimization models. Combinatorial optimization problems, which arise in various areas of human activity [1–6], stand out among discrete modeling problems.

Problems on combinatorial sets are interesting because the feasible region is a combinatorial polyhedron whose properties have been studied. If specific properties of a combinatorial polyhedron are known, they can be used to construct new and perfect the existing methods to solve combinatorial optimization problems.

Note that methods of solving combinatorial problems develop very intensively. Among exact methods, different pruning methods [2, 6] are widely developed; their idea has been proposed first by Danzig and was developed in many other studies, in particular, by Gomory. A group of pruning-plane methods employs the idea of regularization. It consists in immersing the original discrete feasible region in a continuous convex domain, i.e., in rejecting temporarily the discreteness conditions. Then, standard optimization methods are applied to the resultant regular problem. Note that the efficiency of the pruning method is directly dependent on the efficiency of pruning, and this causes certain troubles. Different approaches to pruning and various modifications of the pruning method for combinatorial optimization problems are considered in [2, 6].

Obviously, methods that better and easier describe the properties and specific features of classes of combinatorial problems become most popular. Therefore, developing new methods based on specific properties of combinatorial sets is topical.

The present paper deals with a new method to solve a combinatorial problem taking into account the properties and structure of the set of permutations over which the problem is considered. In particular, the paper describes the construction of a sequence of values of a linear objective function based on expansions of points of the set of permutations on hyperplanes.

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# 1. OPTIMIZATION PROBLEM OVER A COMBINATORIAL SET OF PERMUTATIONS

Let us consider a permutation as an ordered sample of elements

$$a = (a_{i_1}, a_{i_2}, ..., a_{i_n}),$$

(1)

where $a_{i_j} \in A \ \forall i_j \in N_n$, $\forall j \in N_n$, $i_s \neq i_t$, if $s \neq t \ \forall s \in N_n$, $\forall t \in N_n$ from a multiset $A = \{a_1, a_2, ..., a_q\}$ characterized by the basis $S(A) = \{e_1, e_2, ..., e_k\}$, where $e_j \in R^l \ \forall j \in N_k$, and by the multiplicity of elements $k(e_j) = r_j$, $j \in N_k$, $r_1 + r_2 + ... + r_k = q$ according to [4, 5].

The set $E(A)$, whose elements are $n$-samples of the form (1) from the multiset $A$, is called a Euclidean combinatorial set if the conditions $(a' \neq a'') \Leftrightarrow (\exists j \in N_n : a'_j \neq a''_j)$ are satisfied for its arbitrary elements $a' = (a'_1, a'_2, ..., a'_n)$ and $a'' = (a''_1, a''_2, ..., a''_n)$. In other words, two different elements of the set $E(A)$ differ also in the arrangement of their symbols.

The set of permutations with repetition of $n$ real numbers, $k$ of them being different, is called a general set $P_{nk}(A)$, which is a set of ordered $n$-samples (1) from the multiset $A$ provided that $n = q > k$.

If $n = k = q$, we have a set of permutations without repetition. Denote it by $P_n(A)$. Obviously, $P_n(A) = P_{nn}(A)$. If the type of the set of permutations is not specified, we will write $P(A)$. We will consider elements of the set of permutations as points of a Euclidean arithmetic space $R^n$. Then if we specify a mapping $\varphi : E(A) \to E_\varphi(A) \subset R^n$, we have $x_j = a_{i_j}$ $\forall j \in N_n$ for $a = (a_{i_1}, ..., a_{i_n}) \in E(A)$, $x = \varphi(a)$, $x = (x_1 ..., x_n) \in E_\varphi(A)$.

As shown in [4 – 6], the convex shell of a set of permutations is a permutation polyhedron $\Pi(A) = \text{conv} \, P(A)$ whose set of vertices is equal to the set $P(A)$ of permutations: $\text{vert} \, \Pi(A) = P(A)$.

Without loss of generality, let us order elements of the multiset $A$ in nondecreasing order:

$$a_1 \leq a_2 \leq .. \leq a_n,$$

(2)

and elements of its basis, in increasing order: $e_1 < e_2 < ... < e_k$. Then the convex envelope of the general set $P(A)$ of permutations is a general permutation polyhedron $\Pi(A) = \text{conv} \, P(A)$, which can be described by the system of linear inequalities

$$\begin{cases} \sum_{j=1}^{n} x_j \leq \sum_{j=1}^{n} a_j, \\ \sum_{j=1}^{i} x_{\alpha_j} \geq \sum_{j=1}^{i} a_j, \end{cases}$$

(3)

(4)

$\alpha_j \in N_n$, $\alpha_j \neq \alpha_t \ \forall j \neq t$, $\forall j, t \in N_i$, $\forall i \in N_n$, and $P(A) = \text{vert} \, \Pi(A)$.

Let us consider a Euclidean combinatorial optimization problem

$$Z(\Phi, P(A)): \max\{\Phi(a) | a \in P(A)\}$$

that consists in maximizing the function $\Phi(a)$ over the set of permutations $P(A)$, where $\Phi(a) = \sum_{j=1}^{n} c_j x_j$.

If the set $P(A)$ is mapped into the Euclidean space $R^n$, it is possible to formulate a linear programming problem $Z(F, X)$ of maximizing the criterion $F(x)$ over the set $X$, each point $a \in P_{nk}(A)$ being associated with a point $x \in X$ such that $F(x) = \Phi(a)$:

$$Z(F, X): \max\{F(x) | x \in X\},$$

(5)

where $F(x) = \sum_{j=1}^{n} c_j x_j$, $X$ is a nonempty set in $R^n$ defined as $X = \text{vert} \, \Pi(A)$, and $\Pi = \text{conv} \, P(A)$.

Note that it is sometimes expedient to solve a problem

$$x^* = \arg \max_{x \in \Pi(A)} F(x)$$

(6)

for the value of the function $y^* = F(x^*)$. It is also expedient to consider a problem where the value of the objective

function falls within the interval $F(\bar{x}) \le F(x) \le F(\bar{\bar{x}})$. Then problem (6) becomes as follows: determine

$$\bar{x} = \arg\max_{x \in \Pi(A)} F(x) \text{ for } \bar{y} = F(\bar{x}), \qquad \bar{\bar{x}} = \arg\max_{x \in \Pi(A)} F(x) \text{ for } \bar{\bar{y}} = F(\bar{\bar{x}})$$

as $|\bar{x} - \bar{\bar{x}}| \to \min$.

Continuing the studies and developing the results of [4 – 6], we will propose here an approach to solve such problems (problem (6)) by ordering the values of the linear objective function $F(x)$ and constructing the Hamiltonian path for the points at which these values are found. To develop the method, it is necessary to determine the initial point. Let us consider a one-criterion problem without additional constraints.

**Statement 1** [3]. If conditions (2) and the condition

$$c_{i_1} \le c_{i_2} \le .. \le c_{i_n}, \qquad i_n \in N_n, \tag{7}$$

respectively, are satisfied for elements of the multiset $A$ and coefficients of the objective function of the problem

$$\text{extr}\left\{ f(x) = \sum_{j=1}^{n} c_j x_j \,\middle|\, x \in \text{vert } \Pi(A) \right\}$$

then the maximum of the function $f(x)$ over the admissible set is attained at the point $x^* = (x_{i_1}^*, \ldots, x_{i_n}^*) \in \text{vert } \Pi(A)$ specified as

$$x_{i_j}^* = a_j \ \forall j \in N_n,$$

and the minimum is attained at the point $y = (y_1, y_2, \ldots, y_n)$, where

$$y_{i_{j+1}} = a_{n-j} \ \forall j \in N_{n-1} \cup \{0\}.$$

Note that the total number of linear inequalities appearing in the system (3), (4) that describes the permutation polyhedron $\Pi(A)$ is great; therefore, the problem is too complex to be solved by traditional linear programming methods and there is a need for new methods that are based on the properties of the set of admissible solutions and objective functions.

For the problem (6) being considered, the feasible domain includes the permutation polyhedron whose nodes are points of the general set of permutations. Let us formulate some useful properties of the polyhedron.

**THEOREM 1** [4]. The set $P(A)$ lies on the family of $n$-planes of the form

$$\frac{s}{n-s}x_1 + \frac{s}{n-s}x_2 + \ldots + \frac{s}{n-s}x_{n-s} - x_{n-s+1} - \ldots - x_n + a_t^s = 0, \quad t = 1, 2, \ldots, \ \gamma_s \le \frac{n}{s!(n-s)!},$$

and $s$ may take values $1, 2, \ldots, n-1$.

**THEOREM 2** [4, 5]. A point $x = (x_1, x_2, \ldots, x_n)$ is a vertex of the permutation polyhedron $\Pi(A)$ if and only if the following conditions are satisfied:

$$\{\alpha_1^1\} \subset \{\alpha_1^2, \alpha_2^2\} \subset \ldots \subset \{\alpha_1^{n-1}, \ldots, \alpha_{n-1}^{n-1}\} \subset \{\alpha_1^n, \ldots, \alpha_n^n\} = N_n, \quad \sum_{t=1}^{i} x_{\alpha_t^i} = \sum_{t=1}^{i} a_t \ \forall i \in N_n.$$

In applying the method to solve a problem $Z(F, X)$, the theorems formulated above minimize the time it takes to check whether the point found belongs to the combinatorial constraints of the polyhedron and reduce the number of constraints in the original system.

**THEOREM 3** [4]. The vertices of the polyhedron $\Pi(A)$ adjacent to the vertex $\alpha = (a_{i_1}, a_{i_2}, \ldots, a_{i_n})$ have the form $\beta = (a_{j_1}, a_{j_2}, \ldots, a_{j_n})$, where each sequence $(j_1, j_2, \ldots, j_n)$ is obtained from the indices $(i_1, i_2, \ldots, i_n)$ as a result of permutation of indices $i_r$ and $i_t$ such that $|i_r - i_t| = 1$, $a_{i_r} \ne a_{i_t}$.

## 2. METHOD OF ORDERING VALUES OF A LINEAR FUNCTION

As mentioned above, it is required to use the known value of the objective function to determine an extremum point — a vertex of the permutation polyhedron $\Pi(A)$. To this end, we should first find the values of the objective function

at each point, construct a chain (graph) for these values that reflects passages from one point to another, and establish the relationship among them.

Let us use the graph theory. Consider an example of a permutation over the set $A = \{1, 2, 3, 4\}$. The number of elements of the permutation set $P_n(A)$ is 4!, i.e., 24.

There are a lot of methods to generate all the $n!$ permutations of an $n$-element set [7]. In most popular of them, each next permutation is formed from the previous one as follows:

(i) by one transposition;

(ii) by a single transposition of neighboring elements.

The sequences of transpositions obtained using the above methods can be interpreted as a graph $G_n$ whose nodes correspond to all the elements of the set of permutations $P(A)$.

In a graph, two nodes that correspond to permutations $f$ and $g$ are connected by an edge if and only if $g$ can be formed from $f$ by a single transposition of two elements (each vertex is thus connected with exactly $n-1$ other nodes). By Theorem 3, these nodes are adjacent.

As a result, we obtain a complete graph, and any Hamiltonian path corresponds to an alternative generation of all the permutations by a single transposition of two elements at each step. To any vertex of the graph, there corresponds some value of the given function $F(x)$. Of interest is to find a Hamiltonian path along which all values of the function are strictly ordered in decreasing order. In this case, an arbitrary function requires $n!$ its values to be computed and then ordered using $\log_2 n!$ operations. However, this can be done much easier for some functions.

Consider a linear function

$$F(x,c) = (c,x) = \sum_{i=1}^{n} c_i x_i, \tag{8}$$

where $c = (c_1, c_2, \ldots, c_n)$ is a set of arbitrary numbers, $x = (x_1, x_2, \ldots, x_n)$ is a permutation of the numbers $(1, 2, \ldots, n)$. Consider a $u$-permutation from a symmetric group of permutations $S_n$. Let

$$F(x,c,u) = \sum_{i=1}^{n} c_i x_{u(i)}. \tag{9}$$

The basic combinatorial optimization problem is to find a permutation $u_0$ such that $F(x,c,u_0) \leq F(x,c,u)$ for an arbitrary $u \in H$, where $H$ is an arbitrary nonempty subset of the symmetric group $S_n$. If $t \in S_n$, denote $c^t = (c_{t(1)}, c_{t(2)}, \ldots, c_{t(n)})$ and $x^t = (x_{t(1)}, x_{t(2)}, \ldots, x_{t(n)})$. Then $F(x,c) = F(c^t, x^t) = (c,x) = (c^t, x^t)$. Similarly, $F(x,c,u) = F(x^t, c^t, u')$, where $u' = t^{-1}ut$ for any $u$ from $S_n$. Hence, when solving the basic problem, it is always possible to replace a pair $(x,c)$ with a pair $(x^t, c^t)$. In particular, we may always assume that

(a) $c_1 \leq c_2 \leq .. \leq c_n$

or

(b) $c_1 \geq c_2 \geq .. \geq c_n$.

If $H = S_n$, then the problem is solved [8]. It is proved that in case (a), the maximum and minimum of the function $F(x,c,u)$ are achieved for the permutation $u_0 = (1, 2, \ldots, n)$ and $u^* = (n, n-1, \ldots, 2, 1)$, respectively. In what follows, we assume that case (a) always holds for the coefficients $c_i$ $(1 \leq i \leq n)$. Let us now detail the structure of the corresponding graphs for small values of $n$. For $n = 3$, the graph is shown in Fig. 1, where the arc outgoing from the permutation $p_i$ and ingoing to the permutation $p_j$ $(p_i, p_j \in N_6)$ is equivalent to the relationship $F(x,c,p_i) \geq F(x,c,p_j)$.

To make sure that the graph uniquely represents the orderings of values of the functions, it will suffice to calculate

the difference of permutation. For example: $\dfrac{\begin{matrix} 1 & 3 & 2 \\ 1 & 2 & 3 \end{matrix}}{0 \quad 1 \quad -1}$ . The difference corresponds to the difference of the values of the

function $c_2 - c_3 \leq 0$, which is true. This graph has no connections between the permutations 1 3 2 and 2 1 3, and between 2 3 1 and 3 1 2. These pairs are associated with the difference 1 –2 1, which is equivalent to $c_1 - 2c_2 + c_3$. The subtraction sign cannot be defined uniquely here.

Let us now consider the graph for $n = 4$.

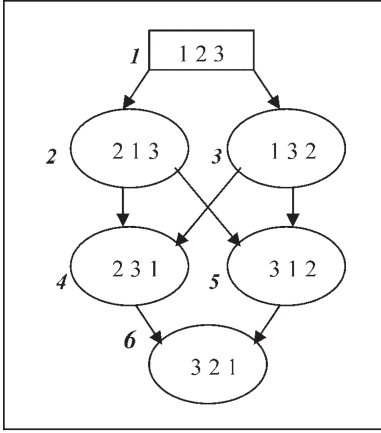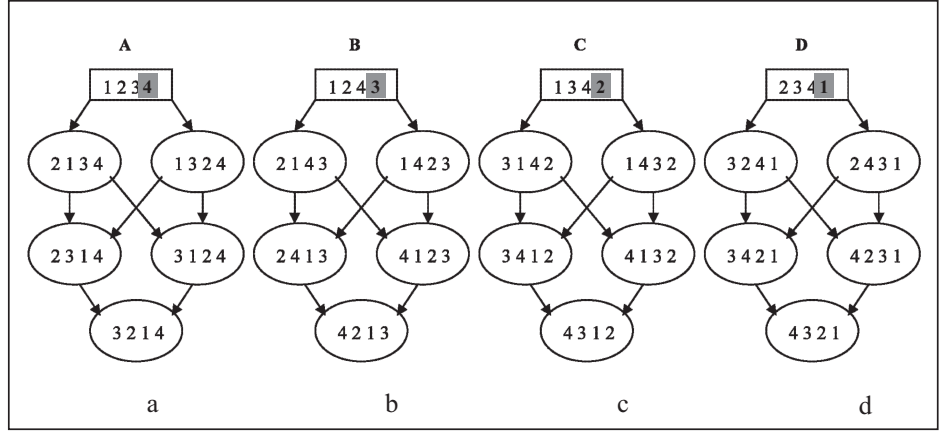Fig. 1                                                                    Fig. 2

**Example 1.** Let there be given a set $A = \{1, 2, 3, 4\}$ used to form the set of permutations $P(A)$. The function

$$F(x) = c_1 x_1 + c_2 x_2 + c_3 x_3 + c_4 x_4$$

is given; its coefficients are ordered as $c_1 \le c_2 \le c_3 \le c_4$ and take on the values $\{1, 2, 3, 4\}$, and $y^* = F(x^*)$ is the value of the function at some point.

It is required to find $x^* = \arg \operatorname{extr} F(x)$, where $x^* \in P(A)$.

**Solution.** As is known [9], $\Pi(A) = \operatorname{conv} P(A)$. Let us expand the graph of the permutation polyhedron $\Pi(A)$ into subgraphs, each fixing one and the same element of the permutation. According to Statement 1 and coefficient-ordering condition, the permutation at which the objective function is maximum is at the uppermost vertex.

Let us fix the permutations where figure 4 is in the fourth place. Since in computing the difference of permutation codes, zero is always in the fourth position, this subgraph is obviously a precise replica of the graph shown in Fig. 1. Figure 2a shows the subgraph for $x_4 = 4$. The graph in Fig. 2b results from uniting the permutations where figure 3 is in the fourth position. Similar graphs are presented in Figs. 2c and 2d, where figures 2 and 1, respectively, are the last in the permutations. The arrows indicate a passage from one point to another, the values of the objective functions being ordered in decreasing order. Note that virtually all the relationships are defined, but relationships among neighboring elements inside (as well as in Fig. 1) should be studied, then it is possible to present the complete Hamiltonian path on the graph of the permutation polyhedron. Thus, it is possible to depict all the subgraphs obtained from the multiset $A$ using one transposition.

Note that the maximum and minimum values of the function are determined uniquely in all the graphs. And all the graphs are a copy of the graph $A$, so that they can be placed in the same order one under another. Note that the graph $B$ can be obtained from the graph $A$, where elements 4 and 3 are transposed in all the permutations. Similarly, the graph $C$ can be obtained from the graph $B$ after the transposition $(3, 2)$, and the graph $D$ can be obtained from the graph $C$ after the transposition $(2, 1)$. Hence, the higher the graph, the greater the value of the function. This can be presented schematically:

$$F(x)_{[A]} \ge F(x)_{[B]} \ge F(x)_{[C]} \ge F(x)_{[D]}.$$

Uniting all the subgraphs yields $\mathbf{G} = \mathbf{A} \cup \mathbf{B} \cup \mathbf{C} \cup \mathbf{D}$. Figure 3 shows the general graph for $n = 4$.

Analyzing Figs. 1–3, we conclude that two incomparable nodes remain in each subgroup. To find the Hamiltonian path with decreasing values of the function, it is necessary to compare values of the function at all these nodes and at the nodes of neighboring subgraphs.

The reasoning above and Figs. 1–3 allow concluding that the points of the set of permutations $P(A)$ can be expanded over parallel hyperplanes in decreasing order of values of the linear objective function $F(x)$ at these points. Expanding points of the combinatorial set of permutations $P(A)$ for $n \ge 4$ provides hierarchical arrangement of these points over the hyperplanes $A$, $B$, $C$, and $D$ (see Fig. 2) according to the values of the objective function $y^* = F(x^*)$.

Let us introduce the following notation: $\Delta_1 = c_2 - c_1$; $\Delta_2 = c_3 - c_2$; and $\Delta_3 = c_4 - c_3$. Relationships among them may depend on their specific values.
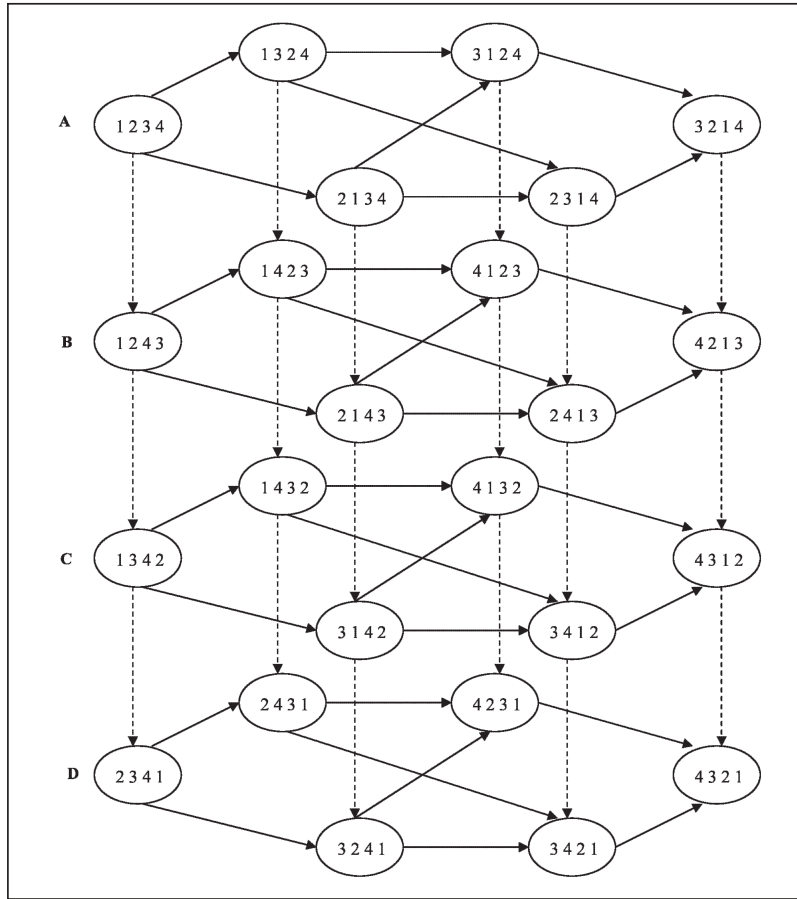
Fig. 3

Let us establish possible relationships for the above notation:

(i) $\Delta_1 = \Delta_2 = \Delta_3$;

(ii) $\Delta_1 > \Delta_2 > \Delta_3$;  (iii) $\Delta_1 > \Delta_3 > \Delta_2$;

(iv) $\Delta_2 > \Delta_1 > \Delta_3$;  (v) $\Delta_2 > \Delta_3 > \Delta_1$;  (10)

(vi) $\Delta_3 > \Delta_1 > \Delta_2$;  (vii) $\Delta_3 > \Delta_2 > \Delta_1$.

Special cases can also be considered:
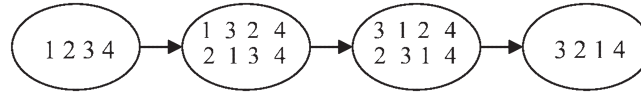
(i) $\Delta_1 = \Delta_2 > \Delta_3$;  (ii) $\Delta_1 = \Delta_2 < \Delta_3$;

(iii) $\Delta_1 = \Delta_3 > \Delta_2$;  (iv) $\Delta_1 = \Delta_3 < \Delta_2$;  (11)

(v) $\Delta_2 = \Delta_3 > \Delta_1$;  (vi) $\Delta_2 = \Delta_3 < \Delta_1$.

To construct the Hamiltonian path, it is necessary to establish relationships on each of the hyperplanes $A$, $B$, $C$, and $D$ (see Fig. 2) among the pairs of points $3, 2$ and $5, 4$ (the points are numbered in Fig. 1).
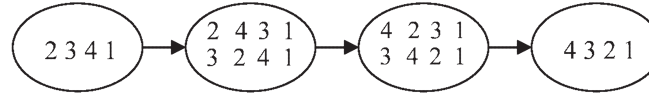
For each case, it is necessary to compute the scheme for the subgraphs $A$, $B$, $C$, and $D$, then to derive the general relationship and specify the Hamiltonian path over the whole permutation polyhedron $\Pi(A)$. As an example, let us consider the first case where $\Delta_1 = \Delta_2 = \Delta_3$. The upper subgraph $A$ obtains two pairs of nodes: $3, 2$, and $5, 4$, where the values of functions coincide and should be equated. Computations yield the relationships

$$\frac{-\begin{matrix} 1 & 3 & 2 & 4 \\ 2 & 1 & 3 & 4 \end{matrix}}{-1+2-1} = \Delta_1 - \Delta_2 = 0, \qquad \frac{-\begin{matrix} 3 & 1 & 2 & 4 \\ 2 & 3 & 1 & 4 \end{matrix}}{+1-2+1} = -\Delta_1 + \Delta_2 = 0.$$

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Then it is possible to depict the following scheme in the subgraph $A$:

$$\boxed{1\,2\,3\,4} \rightarrow \boxed{\begin{matrix}1\,3\,2\,4\\2\,1\,3\,4\end{matrix}} \rightarrow \boxed{\begin{matrix}3\,1\,2\,4\\2\,3\,1\,4\end{matrix}} \rightarrow \boxed{3\,2\,1\,4}$$

The situation is similar in the subgraph $D$:

$$\boxed{2\,3\,4\,1} \rightarrow \boxed{\begin{matrix}2\,4\,3\,1\\3\,2\,4\,1\end{matrix}} \rightarrow \boxed{\begin{matrix}4\,2\,3\,1\\3\,4\,2\,1\end{matrix}} \rightarrow \boxed{4\,3\,2\,1}$$

Considering the subgraph $B$ (see Fig. 2), we obtain the relationship for the following pair of nodes:

$$\dfrac{-\begin{matrix}1\,4\,2\,3\\2\,1\,4\,3\end{matrix}}{-1+3-2}=\Delta_1-2\Delta_2=-\Delta_2<0 \qquad \dfrac{-\begin{matrix}4\,1\,2\,3\\2\,4\,1\,3\end{matrix}}{+2-3+1}=-2\Delta_1+\Delta_2=-\Delta_1<0.$$

The subgraph $B$ has the relations $\Delta_1-2\Delta_2$ and $-2\Delta_1+\Delta_2$. As a result, the Hamiltonian path on $B$ has the form

$$\boxed{1\,2\,4\,3} \rightarrow \boxed{2\,1\,4\,3} \rightarrow \boxed{1\,4\,2\,3} \rightarrow \boxed{2\,4\,1\,3} \rightarrow \boxed{4\,1\,2\,3} \rightarrow \boxed{4\,2\,1\,3}$$
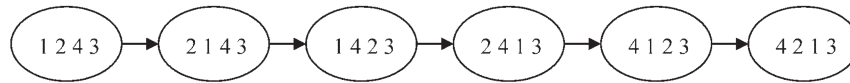
The situation is similar in the subgraph $C$: $1432>3142$ and $4132>3412$; therefore, we have

$$\boxed{1\,3\,4\,2} \rightarrow \boxed{1\,4\,3\,2} \rightarrow \boxed{3\,1\,4\,2} \rightarrow \boxed{4\,1\,3\,2} \rightarrow \boxed{3\,4\,1\,2} \rightarrow \boxed{4\,3\,1\,2}$$
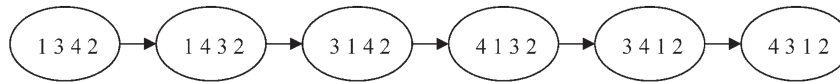
Note that the hyperplanes $A$ and $B$ have points for which the following relationship holds: $\dfrac{-\begin{matrix}1\,3\,2\,4\\1\,2\,4\,3\end{matrix}}{+1-2+1}$
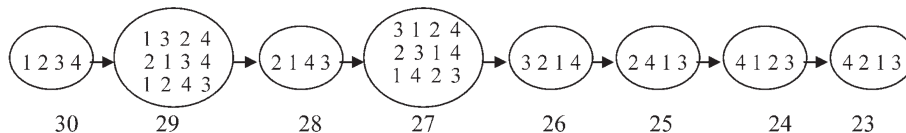$=-\Delta_3+\Delta_4=0$, i.e., the values of the function coincide at these points.

Similarly, the value of the function at the point 3 1 2 4 on the hyperplane $A$ is equal to the value of the function at the point 1 4 2 3 on the hyperplane $B$ since $\dfrac{-\begin{matrix}3\,1\,2\,4\\1\,4\,2\,3\end{matrix}}{+2-3\,0\,1}=-2\Delta_1+\Delta_2+\Delta_3=0,$ and the value of the function at the point 3 2 1 4 on the hyperplane $A$ is greater than the value of the function at the point 2 4 1 3 on the hyperplane $B$ since the condition $\dfrac{-\begin{matrix}3\,2\,1\,4\\2\,4\,1\,3\end{matrix}}{+1-2\,0\,1}=-\Delta_1+\Delta_2+\Delta_3>0$ is satisfied.

Let us calculate the values of the objective function at the following points:

$F(x_1)=1*1+2*2+3*3+4*4=30,$     $F(x_7)=1*2+2*3+3*1+4*4=27,$

$F(x_2)=1*1+2*3+3*2+4*4=29,$     $F(x_8)=1*1+2*4+3*2+4*3=27,$

$F(x_3)=1*2+2*1+3*3+4*4=29,$     $F(x_9)=1*3+2*2+3*1+4*4=26,$

$F(x_4)=1*1+2*2+3*4+4*3=29,$     $F(x_{10})=1*2+2*4+3*1+4*3=25,$

$F(x_5)=1*2+2*1+3*4+4*3=28,$     $F(x_{11})=1*4+2*1+3*2+4*3=24,$

$F(x_6)=1*3+2*1+3*2+4*4=27,$     $F(x_{12})=1*4+2*2+3*1+4*3=23.$

The above reasoning allows arranging nodes of the subgraphs $A$ and $B$ as a chain depending on the values of the objective function:

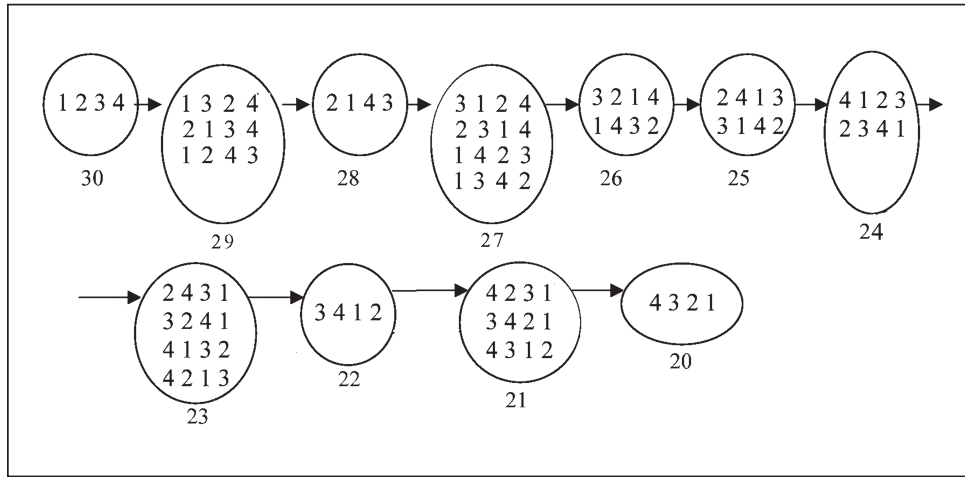$$\boxed{1\,2\,3\,4} \rightarrow \boxed{\begin{matrix}1\,3\,2\,4\\2\,1\,3\,4\\1\,2\,4\,3\end{matrix}} \rightarrow \boxed{2\,1\,4\,3} \rightarrow \boxed{\begin{matrix}3\,1\,2\,4\\2\,3\,1\,4\\1\,4\,2\,3\end{matrix}} \rightarrow \boxed{3\,2\,1\,4} \rightarrow \boxed{2\,4\,1\,3} \rightarrow \boxed{4\,1\,2\,3} \rightarrow \boxed{4\,2\,1\,3}$$

| 30 | 29 | 28 | 27 | 26 | 25 | 24 | 23 |

Fig. 4

Let us consider nodes in the subgraph $C$, in particular, 3 1 4 2 and 1 4 3 2: since $\dfrac{-\begin{matrix}3 & 1 & 4 & 2\\ 1 & 4 & 3 & 2\end{matrix}}{+2\,-3\ 1}=-2\Delta_1+\Delta_2=-\Delta_1<0$, the value of the function at the node 3 1 4 2 is greater than that at the node 1 4 3 2.

The relationships for the values of the function at the nodes 4 1 3 2 and 3 4 1 2 are similar since $\dfrac{-\begin{matrix}4 & 1 & 3 & 2\\ 3 & 4 & 1 & 2\end{matrix}}{+1\,-3\ +2}$ $=-\Delta_1+2\Delta_2=\Delta_2>0$.

The relationship $-\Delta_1+2\Delta_2$ occurs, for which it is necessary to determine the sign.

If we consider a relationship among points from different hyperplanes, the value of the function at the point 1 3 4 2 from $B$ is equal to that at the point 1 4 2 3 from $C$ since $\dfrac{-\begin{matrix}1 & 3 & 4 & 2\\ 1 & 4 & 2 & 3\end{matrix}}{-1\ 2\,-1}=\Delta_2-\Delta_3=0$.

The situation is similar with the points 3 2 1 4 from $A$ and 1 4 3 2 from $B$ since the value of the relation is $\dfrac{-\begin{matrix}3 & 2 & 1 & 4\\ 1 & 4 & 3 & 2\end{matrix}}{+2\,-2\,-2+2}=-2\Delta_1+2\Delta_2=0$.

Let us consider also the points 2 4 1 3 from $B$ and 1 4 3 2 from $C$ and their relationship $\dfrac{-\begin{matrix}2 & 4 & 1 & 3\\ 1 & 4 & 3 & 2\end{matrix}}{+1\ 0\,-2+1}$ $=-\Delta_1+\Delta_3=0$, and the points 2 4 1 3 from $B$ and 3 1 4 2 from $C$; at these points, the values of the function are equal since $\dfrac{-\begin{matrix}2 & 4 & 1 & 3\\ 3 & 1 & 4 & 2\end{matrix}}{-1+3\,-3\ 1}=\Delta_1-2\Delta_2+\Delta_3=0$.

The values of the function are also equal at the points 4 2 1 3 from $B$ and 4 1 3 2 from $C$ since $\dfrac{-\begin{matrix}4 & 2 & 1 & 3\\ 4 & 1 & 3 & 2\end{matrix}}{+1\,-2+1}$ $=-\Delta_2+\Delta_3=0$.

Obviously, the value of the function at the point 2 3 4 1 from $D$ is equal to that at the point 4 1 2 3 from $B$ since $\dfrac{-\begin{matrix}2 & 3 & 4 & 1\\ 4 & 1 & 2 & 3\end{matrix}}{-2+2+2-2}=2\Delta_1-2\Delta_2=0$.
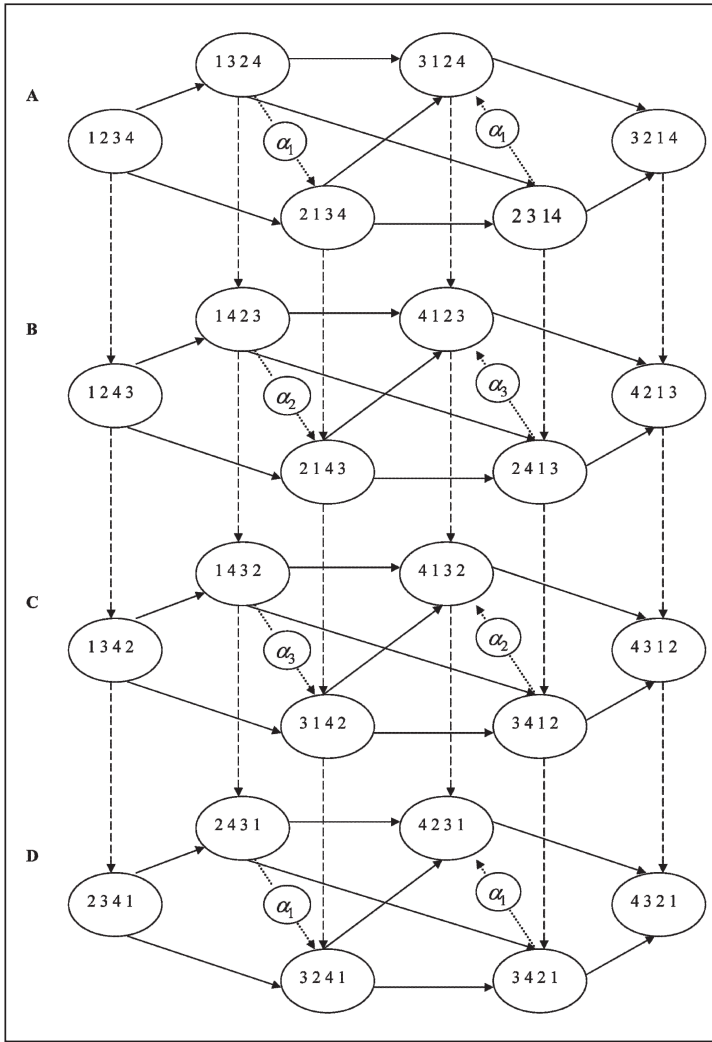
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Fig. 5
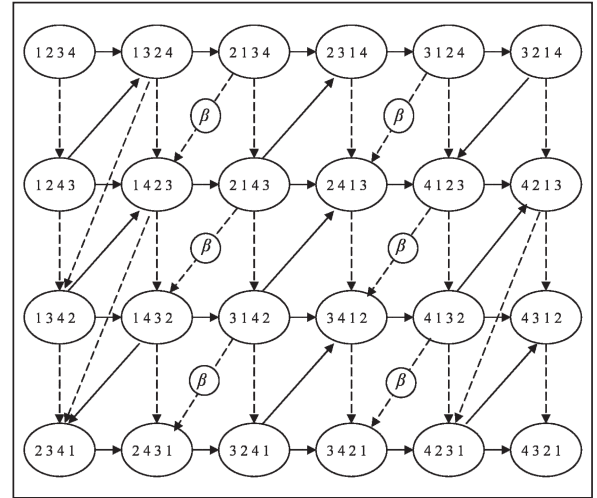


Fig. 6
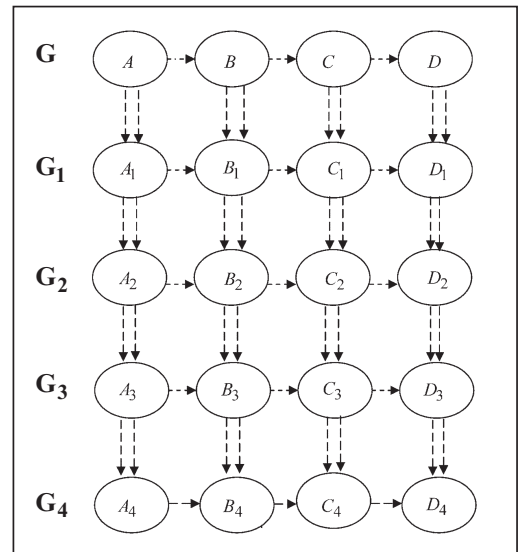


Fig. 7

The situation is similar for the points 4 1 3 2 from **C** and 3 2 4 1 from **D** since
$$\frac{\begin{array}{cccc} -4 & 1 & 3 & 2 \\ 3 & 2 & 4 & 1 \end{array}}{+1 -1 -1 +1} = -\Delta_1 + \Delta_3 = 0, \text{ and}$$

for the points 4 3 1 2 from **C** and 3 4 2 1 from **D** since
$$\frac{\begin{array}{cccc} -4 & 3 & 1 & 2 \\ 3 & 4 & 2 & 1 \end{array}}{+1 -1 -1 +1} = -\Delta_1 + \Delta_3 = 0.$$

As a result of calculations, we obtain the Hamiltonian path of the graph of the permutation polyhedron and ordering of all the values of the linear function $F(x)$ in decreasing order for points of the subgraphs **A**, **B**, **C**, and **D** (Fig. 4).

Other alternatives from (10) can be calculated similarly.

In determining the Hamiltonian path, it is necessary to answer one of three questions: Is a certain expression positive, negative, or zero? Let us call them $\alpha$-questions. The corresponding expressions are presented below (they have already been presented in the previous calculations):

$$\alpha_1 = \Delta_1 - \Delta_2; \ \alpha_2 = \Delta_1 - 2\Delta_2; \ \alpha_3 = 2\Delta_1 - \Delta_2.$$

The general scheme in Fig. 4 is transformed to the scheme in Fig. 5 where the arrow direction remains the same for $\alpha_i > 0$.

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There is a certain relationship between the questions:

(i) if $\alpha_1 > 0$, then $\alpha_2 > 0$ and $\alpha_3$ is not defined,
(ii) if $\alpha_1 < 0$, then $\alpha_2 < 0$ and $\alpha_3$ is not defined;
(iii) if $\alpha_1 = 0$, then $\alpha_2 < 0$ and $\alpha_3 > 0$.

Now, let us consider case (ii) from (10): $\Delta_1 > \Delta_2 > \Delta_3$. In our case, $\alpha_1 > 0$; $\alpha_3 > 0$ and $\alpha_2$ is not defined. If $\alpha_2 > 0$, then the general graph takes the form as in Fig. 6. Six arrows are added, which are denoted as $\beta$-questions. These questions arise in comparing nodes that belong to different neighboring subgraphs. In our case, it is necessary to answer only one question of the form $\beta = -\Delta_1 + 2\Delta_2 + \Delta_3$.

For other answers of $\alpha$-questions, additional $\beta$-questions may arise.

The arcs available are insufficient to construct the Hamiltonian path. It is necessary to compare nodes in neighboring columns, for example, the pairs 2314 and 3142 and 2413 and 3241. We will call questions of such type $\gamma$-questions and consider them in further publications.

The example was considered for the case $n = 4$; however, the results can be generalized for $n = 5$. Then, $5! = 120$ points should be considered in the set of permutations $P(A)$, which are located on five hyperplanes of the form $\mathbf{G}$ (see Fig. 5), each containing 24 points (Fig. 7).

It is necessary to consider $\alpha$-questions inside each subgraph $\mathbf{G}_i$ $(1 \leq i \leq 4)$ presented in Fig. 2. For an arbitrary dimension $n$, the situation is similar, i.e., the permutation polyhedron always contains subgraphs of the form $\mathbf{G}_i$, for which a need always arises to consider $\alpha$- and $\beta$-questions.

## CONCLUSIONS

Complex combinatorial problems over a set of permutations have been analyzed. Some properties of the admissible domain of a Euclidean combinatorial problem with specific input data have been considered. The method of ordering the values of a linear function on a set of permutations has been developed and justified.

Further study will be focused on implementing and adapting the method formulated here and on developing new methods to solve combinatorial optimization problems in view of input data and various combinatorial sets.

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