# CONSTRUCTION OF HAMILTONIAN PATHS IN GRAPHS OF PERMUTATION POLYHEDRA

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*The problem of finding an extremum of a linear function over a permutation set is considered. The polyhedron of admissible values of this function over permutations is constructed. The constructed graph is shown to be partially ordered with respect to the transposition of two elements of a permutation. Based on this property, a method is proposed for the construction of a Hamiltonian path in the graph corresponding to the permutation set for* $n = 4$.

**Keywords:** *polyhedral combinatorics, combinatorial set, combinatorial permutation set, permutation polyhedron, graph, hyperface, Hamiltonian path.*

## INTRODUCTION

Graphs of polyhedra possess many specific properties, and, in investigating them, many problems arise, which are of interest not only for graph theory, combinatorics, and topology but also for linear programming theory. W. Hamilton constructed prime cycles containing each vertex of a dodecahedron. The supposition that each polyhedral graph is Hamiltonian was stated later on. This stipulated the publication of many works devoted to the establishment of the Hamiltonianness of polyhedral graphs. It should be noted that properties of graphs and polyhedra are widely used in investigating many classes of combinatorial models and in developing new methods for solving them [1–10]. Most interesting results are obtained for polyhedra of the packing problem, maximal graph matching problem, travelling salesman problem, knapsack problem, etc. The travelling salesman problem plays an important role in discrete optimization. The admissible domain of this problem is described by a polyhedron of Hamiltonian cycles and circuits. In many sources, the possibility of linearization of this problem, i.e., the construction of the convex envelope of its feasible solutions, is investigated. The use of information on the structure of the convex envelope of feasible solutions that underlies many methods of solution of combinatorial problems is one of most successful approaches to the solution of combinatorial optimization problems to date. The combinatorial theory of polyhedra investigates extremal properties of polyhedra by considering the set of its faces of all dimensionalities as some complex. But, in solving such problems, problems arise that are connected with the complexity of mathematical models, large amount of information, and others since the majority of problems over combinatorial sets are *NP*-complete. Worthy of mention is a close relation of properties of polyhedrons and their graphs to problems of estimation of the number of iterations and efficiency of algorithms of this type in linear programming problems.

The majority of problems over graphs deal with the determination of connectivity components, distances, search for routes, etc. However, graphs of real-world problems are very large, and their analysis is possible only with the use of modern computer facilities. Therefore, the ultimate goal of consideration of such a problem is the description and realization of a practical algorithm for the solution of this problem on a computer. To date, important results are obtained in the field of investigation of various classes of combinatorial models and development of new methods for solving them.

This work continues the investigation of combinatorial problems over various sets of permutations, combinations, and polypermutations considered in [4, 5, 10–13]. In this article, based on the established interrelation between problems over

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combinatorial sets and graphs of polyhedra of corresponding sets, some structural properties of an admissible domain are investigated and also a method based on graphs is formulated that solves a combinatorial problem. In particular, in [4], a method of construction of a Hamiltonian path inside hyperfaces is described. In this article, the problem of construction of a Hamiltonian path between hyperfaces is stated, i.e., a method of construction of a Hamiltonian path is proposed that consists of "pasting" the third layer in two layers if the Hamiltonian path inside the two layers is known.

## 1. MATHEMATICAL PROBLEM STATEMENT

The general combinatorial optimization problem lies in finding an extremum of a linear objective function over a permutation polyhedron under additional linear constraints. As a rule, in solving a class of such problems, the possibility of their linearization, i.e., the construction of the convex envelope of feasible solutions of a problem is investigated. The passage from the parametric form of specification of a convex polyhedron to the analytical form is of great importance for discrete optimization problems since it allows one to formulate them in terms of linear programming, but, as practice implies, it is not always justified. A subproblem of the above problem can be the determination of the Hamiltonian path that reflects a change in the value of the objective function over a permutation set. The polyhedron of Hamiltonian cycles of a graph is a face of the polyhedron of Hamiltonian cycles of the complete graph. Let us consider the well-known graph of the permutation polyhedron from [4]. Let us mention the main property of this graph. It reflects a partial ordering of the permutation set for $n=4$ with respect to values of an arbitrary linear function $f(x)=c_1x_1+c_2x_2+c_3x_3+c_4x_4$, where $c_1 \le c_2 \le c_3 \le c_4$ and the collection $x=(x_1,x_2,x_3,x_4)$ goes through the set of all permutations $p_n$. In a graph, two vertices corresponding to two permutations are adjacent if they differ from one another in positions of only two elements. In other words, two permutations are adjacent if they are obtained from one another with the help of transposition of two elements.

**LEMMA 1.** For two adjacent permutations, the value of the function $f(x)$ is no smaller (no larger) for the permutation in which the maximal element out of two different elements is on the right.

This lemma is true for an arbitrary *n*. In fact, let $p_1=(x_1,x_2,\ldots,x_k,\ldots,x_l,\ldots,x_n)$ and $p_2=(x_1,x_2,\ldots,x_l,\ldots,x_k,\ldots,x_n)$ be two permutations that differ in the position of two elements $x_k$ and $x_l$ and, at the same time, let $x_l > x_k$. We consider the difference of values $f(p_1)-f(p_2)$. After simplification, we obtain that it is equal to $c_k(x_k-x_l)+c_l(x_l-x_k)=(c_l-c_k)(x_l-x_k)$. Since, for $l<k$, we always have $c_l \ge c_k$, this expression is no smaller than zero, which substantiates the truth of the lemma.

In the graph presented in Fig. 1, all adjacent permutations are connected by arcs according to Lemma 1. The graph of the permutation polyhedron $G(P_n)$ for an arbitrary $n$ is similarly constructed.

**COROLLARY.** The linear function $f(x)$ on a permutation polyhedron $G(P_n)$ assumes the maximal value at the permutation $(1,2,\ldots,n)$ and the minimal value at the permutation $(n,n-1,\ldots,2,1)$.

In fact, at any $j$th position of any permutation distinct from the first one, a number smaller than $j$ is located. By Lemma 1, this means that the function value at this permutation is always smaller than the initial one, which is what had to be proved. The same reasoning can be repeated for the minimal function value.

For such graphs, the following problem arises rather often: find the set of permutations at which the value of the objective function is equal to a given value, i.e.,

$$x^* = \arg_{x \in P_n} f(x), \tag{1}$$

where $f(x^*)=y$. It also makes sense to consider a similar problem in which permutations for which the objective function assumes a given value do not always exist. Then the above problem is formulated as follows: determine the set of pairs of permutations $(\underline{x},\overline{x})$ for which, for a given $y$, we have

$$\begin{aligned} \overline{x} &= \arg\min_{f(x)>y} f(x), \\ \underline{x} &= \arg\max_{f(x)<y} f(x). \end{aligned} \tag{2}$$

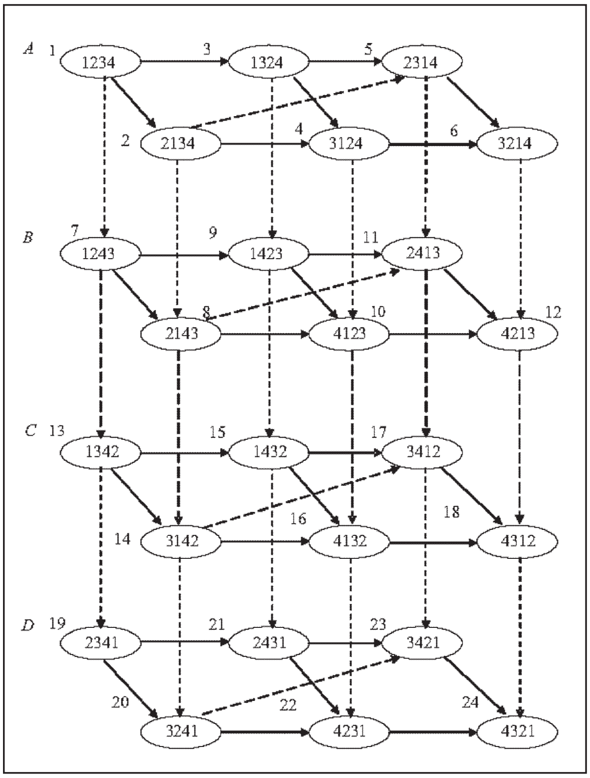


Fig. 1

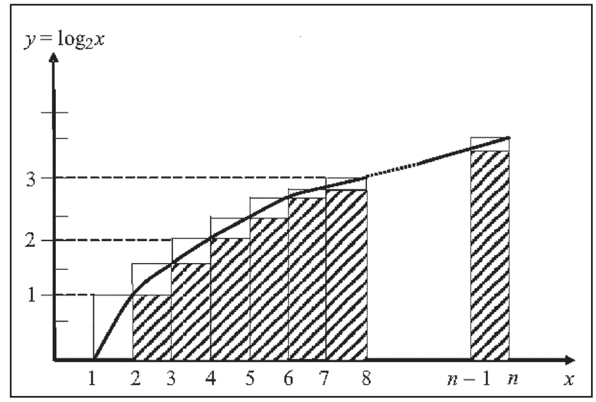


Fig. 2

## 2. AN APPROACH TO THE SOLUTION

It is obvious that both problems can be solved if, in a graph $G(P_n)$, the set of arcs between nonadjacent permutations is extended so that it is possible to traverse a path along arcs from the initial vertex at which $f(x)$ assumes the maximal value to the terminal vertex at which $f(x)$ assumes its minimal value. In this case, it is necessary to traverse all the other graph vertices. It is this path that is called Hamiltonian. If the sequence of the permutations being traversed is known, then, with the help of the dichotomy of the Hamiltonian path, computing the values of the function at the corresponding permutation, one can always localize an arbitrary value of the objective function $f(x)$.

**LEMMA 2.** The upper estimate of the complexity of solving problems (1) and (2) is a polynomial whose degree does not exceed two.

Since the number of vertices of a graph $G(P_n)$ (permutations) equals $n!$, the complexity of computations providing the dichotomy is estimated by the quantity $R = \log_2 n! = \sum_{i=2}^{n} \log_2 i$. To determine it, we consider the plotted function $y = \log_2 x$ in Fig. 2.

The area of all rectangles that are over the curve of $y$ and whose bases are located on the axis $Ox$ is equal to $\overline{S} = \sum_{i=2}^{n} \log_2 i = R$. Similarly, the area of all (shaded) rectangles constructed under the curve of $y$ is equal to $\underline{S} = \sum_{i=2}^{n-1} \log_2 i = R - \log_2 n$. As is seen from the figure, the area bounded by the curve $y = \log_2 x$ and the abscissa axis satisfies the constraints

$$R - \log_2 n < \int_1^n \log_2 x dx < R. \tag{3}$$

The value of the indefinite integral equals $[x \ln x - x] / \ln 2$. Since $n >> \log_2 n$, we obtain the estimate $R \le n^2$, which is what had to be proved.

We pass directly to the construction of a Hamiltonian path in the graph $G(P_4)$. For the convenience of manipulating by permutations, we number them from 1 to 24 as is shown in Fig. 1 and call them points (vertices) $p_i$ $(i = 1, 2, \dots, 24)$, for example, $p_{16} = (4, 1, 3, 2)$. We denote a vector of coefficients of the function $f(x)$ by $c = (c_1, c_2, \dots, c_n)$. In Fig. 1, we have $c = (c_1, c_2, c_3, c_4)$. Then the value of the function $f(x)$ at an arbitrary point $p_i$ $(1 \le i \le 24)$ is found as the scalar product $f(p_i) = (p_i, c)$. Let us show that the graph $G(P_4)$ can be inductively constructed beginning with the first two permutations. Assume that the vertices $p_1$ and $p_2$ form a subgraph of the graph $G(P_4)$ in which the last two elements, 3 and 4, are fixed. The vertex $p_2$ is constructed from $p_1$ by the transposition of the elements 1 and 2 and, hence, according to Lemma 1, we have $f(p_1) \ge f(p_2)$. Now, changing over the elements 2 and 3 in the vertices $p_1$ and $p_2$, we obtain the vertices $p_3$ and $p_4$ in which the relation $f(p_3) \ge f(p_4)$ remains true, and, moreover, by Lemma 1, we have $f(p_1) \ge f(p_3)$ and $f(p_2) \ge f(p_4)$. Similarly, transposing the elements 1 and 2 at the vertices $p_3$ and $p_4$, we obtain the vertices $p_5$ and $p_6$. As a result of these actions, we obtain a subgraph $A$ that contains all the permutations of $P_4$ with the fixed fourth element $x_4 = 4$. It is obvious that, in the subgraph $A$, $f(x)$ assumes its maximal value at the vertex $p_1$ and its minimal value at the vertex $p_6$. However, the number of arcs in this subgraph is insufficient for the construction of a Hamiltonian path. It is possible to construct one more arc from the vertex $p_2$ to the vertex $p_5$ since these permutations differ from one another in the transposition of the numbers 1 and 3, but this is also not sufficient (this arc is labeled by a dotted line).

We take all the permutations of the subgraph $A$ and simultaneously transpose the numbers 3 and 4 in them. As a result, we obtain the subgraph $B$ that contains all the same permutations of $P_4$ in which the fourth element $x_4 = 3$ is fixed. By Lemma 1, the corresponding vertices of the subgraphs $A$ and $B$ are adjacent and the arcs connecting them traverse from the top (from the subgraph $A$) downward (to the subgraph $B$). It is obvious that the internal orientation of the subgraph $B$ preserves the internal orientation of the subgraph $A$. Now, if the numbers 2 and 3 are transposed in all permutations in the subgraph $B$, then, as a result, we obtain the subgraph $C$ containing all the permutations of $P_4$ in which the fourth element $x_4 = 2$ is fixed. It is obvious that this subgraph also preserves the internal orientation of the subgraph $B$ (and also of the subgraph $A$) and each its vertex contains an arc from the corresponding vertex of the subgraph $B$. And, finally, if the numbers 1 and 2 are transposed in each permutation in the subgraph $C$, then we obtain the subgraph $D$ that contains all the permutations of $P_4$ with the fixed fourth element $x_4 = 1$.

All the four subgraphs $A, B, C$, and $D$ form the graph $G(P_4)$. One can say that it is inductively constructed beginning with the two vertices $p_1$ and $p_2$. These vertices have first been supposedly twice projected sequentially and have formed the subgraph $A$. Then the subgraph $A$ has been triply projected and has form the entire graph $G(P_4)$.

Let us return to the question of the construction of a Hamiltonian path in the subgraph $A$. It is fundamentally important to establish the relationship between values of functions at the points $p_2$ and $p_3$ and also at the points $p_4$ and $p_5$. Let us denote the corresponding differences by $\delta f$ and consider them,

$$\begin{aligned} \delta f(2,3) &= f(p_2) - f(p_3) = (p_2, c) - (p_3, c) = c_1 - 2c_2 + c_3; \\ \delta f(4,5) &= f(p_4) - f(p_5) = (p_4, c) - (p_5, c) = c_1 - 2c_2 + c_3. \end{aligned} \quad (4)$$

They coincide and, hence, if the value of $\delta f(2,3)$ is known, then this fact is sufficient to determine a Hamiltonian path in the subgraph $A$. If $\delta f(2,3) \ge 0$, then we obtain the arcs $p_2 p_3$ and $p_4 p_5$, which determines the Hamiltonian path $\sigma_1(A) = (1, 2, 3, 4, 5, 6)$. If $\delta f(2,3) < 0$, then we obtain the arcs $p_3 p_2$ and $p_5 p_4$, which gives the Hamiltonian path $\sigma_2(A) = (1, 3, 2, 5, 4, 6)$.

Since the first three elements in the permutations vary in projecting the subgraph $A$ onto the subgraphs $B$, $C$, and $D$, relationships (4) that hold true for the subgraph $A$ are not extended to the subsequent subgraphs. We introduce the following denotations: $\Delta_i = c_{i+1} - c_i$ $(i = 1, 2, \dots, n-1)$. Let $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_{n-2})$ be a vector reflecting distinctive features of the given linear function $\Delta_i = \alpha_{i-1} \Delta_{i-1}$ $(i = 2, 3, \dots, n-1)$. In each subgraph $A, B, C$, and $D$, the fourth element of all permutations is fixed and, hence, the structure of these subgraphs depends only on the first three elements; for the general case, we denote them by $(i_1, i_2, i_3)$, where $(i_1 < i_2 < i_3)$. In each subgraph, these elements determine the first vertex $P_j$, where $j \equiv 1 \pmod 6$, i.e., the vertices $p_1$, $p_7$, $p_{13}$, and $p_{19}$. If, for an arbitrary $n$, we choose a subgraph from $G(P_n)$ in the permutations of which the last $n-4$ elements $(i_5, i_6, \dots, i_{n-1}, i_n) \in N_n$ are fixed, then we obtain a subgraph of the type $G(P_4)$ in which the elements $i_1 < i_2 < i_3 < i_4$ will be instead of elements 1–4. The fourth element $i_4$ in the graph $A$, the element $i_3$ in the graph $B$, the element $i_2$ in the graph $C$, and the element $i_1$ in the graph $D$ will be fixed.

**Definition.** We call the structural coefficients of the graph $G(P_4)$ the quantities

$$\lambda_A = \frac{i_2 - i_1}{i_3 - i_2};\ \lambda_B = \frac{i_2 - i_1}{i_4 - i_2};\ \lambda_C = \frac{i_3 - i_1}{i_4 - i_3};\ \lambda_D = \frac{i_3 - i_2}{i_4 - i_3}. \tag{5}$$

To construct a Hamiltonian path in each subgraph, it is necessary to compute the difference between the values of the function at vertices $p_i$ and $p_j$, where $i \equiv 2 (\mathrm{mod}\ 6)$ and $j \equiv 3 (\mathrm{mod}\ 6)$, and also at vertices $p_k$ and $p_l$, where $k \equiv 4 (\mathrm{mod}\ 6)$ and $l \equiv 5 (\mathrm{mod}\ 6)$. In the subgraph *A*, we have $p_i = (i_2, i_1, i_3, i_4)$ and $p_j = (i_1, i_3, i_2, i_4)$ and, hence, we obtain

$$\begin{gathered} \delta f(2,3) = (p_i, c) - (p_j, c) = c_1(i_2 - i_1) - c_2(i_3 - i_1) + c_3(i_3 - i_2), \\ \delta f(2,3) = -(i_2 - i_1)\Delta_1 + (i_3 - i_2)\Delta_2. \end{gathered} \tag{6}$$

The direction of the arc connecting the vertices $p_2$ and $p_3$ is $p_2 p_3$ if we have $\delta f(2,3) \geq 0$, and this is possible only under the following condition:

$$\frac{\Delta_2}{\Delta_1} \geq \frac{i_2 - i_1}{i_3 - i_2} \text{ or } \alpha_1 \geq 1/\lambda_A. \tag{7}$$

Continuing the same computations for the vertices $p_4$ and $p_5$, we obtain $\delta f(4,5) = -(i_3 - i_2)\Delta_1 + (i_2 - i_1)\Delta_2$. The direction of the arc is $p_4 p_5$ under the conditions

$$\frac{\Delta_2}{\Delta_1} \geq \frac{i_3 - i_2}{i_2 - i_1} \text{ or } \alpha_1 \geq 1/\lambda_A. \tag{8}$$

Similar results can also be obtained for the other subgraphs. For example, for the subgraph *C*, we compute $\delta f(14,15)$. The direction of the arc is $p_{14} p_{15}$ if

$$\frac{\Delta_2}{\Delta_1} \geq \frac{i_3 - i_1}{i_4 - i_3} \text{ or } \alpha_1 \geq \lambda_C,$$

and the arc $p_{16} p_{17}$ takes place when $\alpha_1 \geq 1/\lambda_C$.

Let us consider numbers of the vertices of the graph $G(P_4)$ in the residue class mod 6, where the system (1,2,3,4,5,6) is taken as the complete residue system. The latter computations allow us to assert that the theorem formulated below is true.

**THEOREM 1.** In an arbitrary subgraph *X* of the graph $G(P_4)$, where $X \in \{A, B, C, D\}$, a Hamiltonian path passes as follows:

$$\begin{gathered} (1,3,2,5,4,6)(\mathrm{mod}\ 6) \text{ if } \alpha_1 \leq \min(\lambda_x, 1/\lambda_x); \\ (1,2,3,4,5,6)(\mathrm{mod}\ 6) \text{ if } \alpha_1 \geq \max(\lambda_x, 1/\lambda_x); \\ (1,2,3,4,5,6)(\mathrm{mod}\ 6) \text{ if } 1/\lambda_x \leq \alpha_1 \leq \lambda_x \text{ и } \lambda_x \geq 1; \\ (1,2,3,5,4,6)(\mathrm{mod}\ 6) \text{ if } \lambda_x \leq \alpha_1 \leq 1/\lambda_x \text{ and } \lambda_x \leq 1. \end{gathered} \tag{9}$$

All four Hamiltonian paths can take place under one condition, namely, if, for $\alpha_1$, the equality is not satisfied in any of constraints (9). If an equality of type (7) is satisfied, then we have $f(p_{i)}) = f(p_j)$, the edge is $p_i p_j$, and it can be traversed in an arbitrary direction, which implies that the first and fourth paths (or the second and third paths) are equivalent. If an equality of type (8) is satisfied, then the same also holds true for the edge $p_i p_j$, where $i \equiv 4(\mathrm{mod}\ 6)$ and $j \equiv 5(\mathrm{mod}\ 6)$, and then the first and third paths (or the second and fourth paths) are equivalent. A special case arises when $\lambda_x = 1$. If $\alpha_1 \neq 1$, then only the first two Hamiltonian paths are possible. If $\alpha_1 = 1$, then all the four paths are equivalent since the mentioned two pairs of vertices can be combined to obtain two vertices, and the Hamiltonian paths are written in the general form as follows: $[1,(2,3),(4,5),6]\,(\mathrm{mod}\ 6)$. In order to prevent such indefinitenesses, we will attach the values of $\alpha_1$ that are equal to some parameter $\lambda_x$ (or $1/\lambda_x$) to the left interval. Thereby, we obtain that, in an arbitrary case, one Hamiltonian path will be chosen (out of four possible paths) for each subgraph *A*,*B*,*C*, and *D*. Does this mean that, for the construction of a Hamiltonian path in the graph $G(P_4)$, one should consider $4^4$ variants of different combinations of these paths? Let us consider this question with allowance for the statement formulated below.

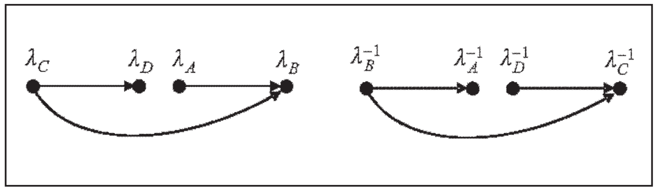


Fig. 3

TABLE 1

| Values of $\alpha_1$ | Results of Computations of Hamiltonian paths along Vertices in Subgraphs | | | |
|---|---|---|---|---|
| | *A* | *B* | *C* | *D* |
| $\alpha_1 < 1/7$ | 1, 3, 2, 5, 4, 6 | 7, 9, 8, 11, 10, 12 | 13, 15, 14, 17, 16, 18 | 19, 21, 20, 23, 22, 24 |
| $1/7 < \alpha_1 < 2/5$ | 1, 3, 2, 5, 4, 6 | 7, 8, 9, 11, 10, 12 | 13, 15, 14, 17, 16, 18 | 19, 21, 20, 23, 22, 24 |
| $2/5 < \alpha_1 < 1/2$ | 1, 3, 2, 5, 4, 6 | 7, 8, 9, 11, 10, 12 | 13, 15, 14, 17, 16, 18 | 19, 20, 21, 23, 22, 24 |
| $1/2 < \alpha_1 < 3/5$ | 1, 2, 3, 5, 4, 6 | 7, 8, 9, 11, 10, 12 | 13, 15, 14, 17, 16, 18 | 9, 20, 21, 23, 22, 24 |
| $3/5 < \alpha_1 < 5/3$ | 1, 2, 3, 5, 4, 6 | 7, 8, 9, 11, 10, 12 | 13, 15, 14, 17, 16, 18 | 9, 20, 21, 23, 22, 24 |
| $5/3 < \alpha_1 < 5/2$ | 1, 2, 3, 5, 4, 6 | 7, 8, 9, 11, 10, 12 | 13, 14, 15, 17, 16, 18 | 9, 20, 21, 23, 22, 24 |
| $5/2 < \alpha_1 < 2$ | 1, 2, 3, 5, 4, 6 | 7, 8, 9, 11, 10, 12 | 13, 14, 15, 16, 17, 18 | 9, 20, 21, 23, 22, 24 |
| $2 < \alpha_1 < 7$ | 1, 2, 3, 5, 4, 6 | 7, 8, 9, 11, 10, 12 | 13, 14, 15, 16, 17, 18 | 9, 20, 21, 23, 22, 24 |
| $7 < \alpha_1$ | 1, 2, 3, 5, 4, 6 | 7, 8, 9, 11, 10, 12 | 13, 14, 15, 16, 17, 18 | 9, 20, 21, 23, 22, 24 |

**LEMMA 3.** For the structural coefficients of the subgraphs $A, B, C$, and $D$, the following relationships are true:

$$\text{(a) } \lambda_A > \lambda_B;\ \lambda_C > \lambda_B;\ \lambda_C > \lambda_D;$$

$$\text{(b) } 1/\lambda_A < 1/\lambda_B;\ 1/\lambda_C < 1/\lambda_B;\ 1/\lambda_C < 1/\lambda_D.$$

It is obvious that item (b) follows from item (a) and, hence, it suffices to prove the truth of item (a). The first inequality follows from the fact that two quantities have identical numerators, and the denominator of $\lambda_B$ is larger since we have $i_4 > i_3$. The second inequality is true since the numerator of $\lambda_C$ is larger and the denominator is smaller than that of $\lambda_B$. The third inequality follows from the fact that the denominators of two quantities are identical, and the numerator of $\lambda_C$ is larger since $i_3 - i_1$ is larger than $i_2 - i_1$.

**THEOREM 2.** There are no more than nine consistent variants of construction of Hamiltonian paths in the subgraphs $A, B, C$, and $D$ of the permutation graph $G(P_4)$.

Lemma 2 allows one to establish a partial ordering for the values of $\lambda_x$ and $1/\lambda_x$, where $X \in \{A, B, C, D\}$.

This dependence is reflected in Fig. 3.

If any two of these eight parameters are not equal to one another, then their different values on the numerical axis form nine intervals, and one of them contains a concrete value of $\alpha_1$. If some of these parameters coincide, the number of intervals decreases. It is this fact that proves the theorem. Let us consider an example.

**Example.** Let we have $i_1 = 1$, $i_2 = 2$, $i_3 = 4$, and $i_4 = 9$. We obtain the following values: $\lambda_A = 1/2$, $\lambda_B = 1/7$, $\lambda_C = 3/5$, $\lambda_D = 2/5$, $1/\lambda_A = 2$, $1/\lambda_B = 7$, $1/\lambda_C = 5/3$, and $1/\lambda_D = 5/2$. After ordering, we obtain the increasing sequence of points on the numerical axis (1/ 7, 2/ 5, 1/ 2, 3/ 5, 5/ 3, 5/ 2, 2, 7). The results of calculations are presented in Table 1.

It is easy to see that all Hamiltonian paths in subgraphs are determined uniquely if only one parameter, namely, $\alpha_1$ is known. To construct a Hamiltonian path over the entire graph $G(P_4)$, it is necessary to use the second parameter $\alpha_2$. Depending on combinations of this parameter with values of structural coefficients of subgraphs, various variants of Hamiltonian paths in the graph $G(P_4)$ are obtained. If these variants are known, then problems over permutations for $n > 4$ can be reduced to subproblems over subgraphs with a number of vertices $n = 4$.

## CONCLUSIONS

The admitted region of some combinatorial optimization problem is investigated with the use of graph theory. A Hamiltonian path is constructed for an arbitrary polyhedron of permutations that is a subgraph of the graph $G(P_4)$. Possibilities of using these results for the construction of an efficient method for the solution of complicated combinatorial problems over a permutation set are substantiated. The further extension of this work is directed toward the development of new methods for solving combinatorial optimization problems with allowance for input data and other combinatorial sets and also their polyhedra and graphs of polyhedra.

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